

Constant mean curvature hypersurfaces with single valued projections on planar domains

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Abstract

A classical problem in constant mean curvature hypersurface theory is, for given $H \geq 0$, to determine whether a compact submanifold Γ^{n-1} of codimension two in Euclidean space \mathbb{R}_+^{n+1} , having a single valued orthogonal projection on \mathbb{R}^n , is the boundary of a graph with constant mean curvature H over a domain in \mathbb{R}^n . A well known result of Serrin gives a sufficient condition, namely, Γ is contained in a right cylinder C orthogonal to \mathbb{R}^n with inner mean curvature $H_C \geq H$. In this paper, we prove existence and uniqueness if the orthogonal projection L^{n-1} of Γ on \mathbb{R}^n has mean curvature $H_L \geq -H$ and Γ is contained in a cone K with basis in \mathbb{R}^n enclosing a domain in \mathbb{R}^n containing L such that the mean curvature of K satisfies $H_K \geq H$. Our condition reduces to Serrin's when the vertex of the cone is infinite.

A classical problem in the theory of constant mean curvature (CMC) hypersurfaces is to determine whether an $(n-1)$ -dimensional compact embedded connected submanifold Γ^{n-1} of Euclidean space $\mathbb{R}_+^{n+1} = \{x_{n+1} \geq 0\}$ with a single valued orthogonal projection γ^{n-1} on the hyperplane $\mathbb{R}^n = \{x_{n+1} = 0\}$ is the boundary of an n -dimensional graph with given constant mean curvature $H \geq 0$ (called H -graph) over the domain enclosed by the submanifold γ .

In the minimal case, it was shown by Finn [5], [6] for surfaces and then by Jenkins-Serrin [8] and Bakel'man [1], [2] for any dimension that there exists a unique H -graph if the projection of Γ onto \mathbb{R}^n bounds a convex domain. For $H > 0$, it was proved by Serrin [9] that a unique H -graph exists if the mean curvature H_C of the right cylinder $C(\Gamma)$ over Γ orthogonal to \mathbb{R}^n satisfies $H_C \geq H$ (considering the non-normalized mean curvature taken with respect to the normal direction pointing to the simply connected component of $\mathbb{R}^{n+1} \setminus C(\Gamma)$).

We observe that the H -convexity assumption of the domain in the above results can not be dispensed. For instance, it follows from the maximum principle that a circle

in \mathbb{R}^2 with radius strictly larger than $1/H$ cannot be the boundary of an H -graph. Nevertheless, the H -convexity hypothesis may be weakened. In fact, here we prove that Serrin's assumption is a special case of a more general condition that applies to a large class of domains and boundary data.

To state our results, we first introduce some terminology. Let $V \in \mathbb{R}_+^{n+1}$ be a point and γ^{n-1} a compact smooth embedded submanifold of \mathbb{R}^n . We denote by D_γ the closure of the domain enclosed by γ and by $K_V(\gamma)$ the cone with base γ and vertex V , i.e.,

$$K_V(\gamma) = \{tV + (1-t)p : p \in \gamma \text{ and } t \in [0, 1]\}.$$

We refer to the right vertical cylinder

$$C(\gamma) = \{p + te_{n+1} : p \in \gamma \text{ and } t \in [0, +\infty)\}$$

as the cone $K_\infty(\gamma)$ over γ with vertex at infinity. Then, we say that $K_V(\gamma)$ is an H -cone for $V \in \mathbb{R}_+^{n+1} \cup \{\infty\}$ if either $K_V(\gamma) \setminus \{V\}$ or $K_\infty(\gamma)$ is a smooth hypersurface having (non-normalized) mean curvature $H_K \geq H > 0$ with respect to the inner orientation.

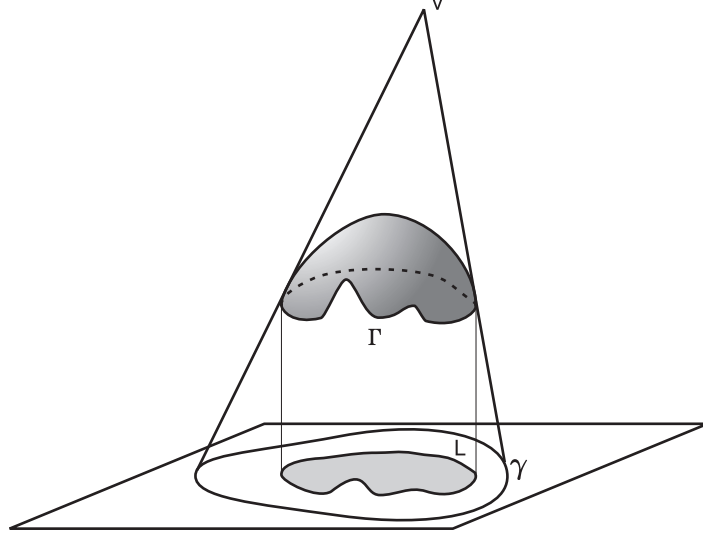
Serrin's result may now be restated as follows: If $K_\infty(\gamma)$ is an H -cone and Γ a compact embedded hypersurface of $K_\infty(\gamma)$ having a single projection onto γ , then there is a unique H -graph with boundary Γ over the domain enclosed by γ . Here, we show that if $\Gamma \subset K_V(\gamma)$ is a compact embedded hypersurface in an H -cone having a single orthogonal projection onto a hypersurface $L \subset D_\gamma$ with mean curvature $H_L \geq -H$, then there is a constant mean curvature H -graph with boundary Γ over the domain enclosed by L .

We treat separately the cases where Γ is smooth ($\mathcal{C}^{2,\alpha}$) or only continuous and obtain results unifying the two statements above in both cases. We first state the result for the smooth case.

Theorem 1. *Let $\Gamma^{n-1} \subset \mathbb{R}_+^{n+1}$ be a compact embedded connected $\mathcal{C}^{2,\alpha}$ submanifold having a single orthogonal projection onto a hypersurface $L^{n-1} \subset \mathbb{R}^n$ such that $H_L \geq -H$ for some constant $H > 0$. If there is an H -cone $K_V(\gamma) \subset \mathbb{R}_+^{n+1}$ such that $\Gamma \subset K_V(\gamma)$ and $L \subset D_\gamma$ if V is finite, then there is a unique H -graph of class $\mathcal{C}^{2,\alpha}$ with boundary Γ over the domain enclosed by L .*

In the case of continuous boundary we have the following result.

Theorem 2. *Let $\Gamma^{n-1} \subset \mathbb{R}_+^{n+1}$ be a compact embedded connected \mathcal{C}^0 submanifold having a single orthogonal projection onto a \mathcal{C}^2 hypersurface $L^{n-1} \subset \mathbb{R}^n$ such that $H_L \geq -H$ for some constant $H > 0$. If there is an H -cone $K_V(\gamma) \subset \mathbb{R}_+^{n+1}$ such that $\Gamma \subset K_V(\gamma)$ and $L \subset D_\gamma$ if V is finite, then there is a unique \mathcal{C}^0 H -graph with boundary Γ which is \mathcal{C}^∞ in the interior.*



From the above, we conclude the existence and uniqueness of H -graphs for a large class of prescribed boundary data over domains of \mathbb{R}^n that are not necessarily mean convex. In fact, let $\gamma \subset \mathbb{R}^n$ be a smooth compact embedded hypersurface satisfying $H_\gamma > H > 0$. Then, let L be a smooth compact embedded hypersurface contained in the interior of the bounded connected component of $\mathbb{R}^n \setminus \gamma$ with mean curvature $H_L \geq -H$. Now, fix a point P in the bounded connected component of $\mathbb{R}^n \setminus L$ and set $\Gamma_t = K_{P+te}(\gamma) \cap C(L)$. Clearly, there is $t_0 > 0$ such that $K_{P+te}(\gamma)$ is an H -cone for any $t \geq t_0$. Therefore, there exists a unique graph over the domain enclosed by L with constant mean curvature H and boundary Γ_t .

Since a H -graph of CMC is given by a solution of a quasi-linear elliptic second order PDE, the existence problem for H -graphs is equivalent to the solvability of a Dirichlet problem for the CMC equation. Consequently, the existence of solutions is usually proved within the theory of elliptic PDE, as is the case of the results stated above.

Proof of Theorem 1. We first consider the case of V finite. The submanifold Γ divides $K_V(\gamma)$ into two connected components. We smooth out the vertex of $K_V(\gamma)$ if V belongs to the simply connected component that we call G . Then G is a graph over the domain Ω enclosed by L of a function $\psi \in C^{2,\alpha}(\bar{\Omega})$. The mean curvature function $H_\psi(x)$ of G at $(x, \psi(x))$ belongs to $C^{1,\alpha}(\bar{\Omega})$ and satisfies $H_\psi(x) \geq H$.

Let $H_t \in C^{1,\alpha}(\bar{\Omega})$, $t \in [0, 1]$, be the family of functions

$$H_t = (1 - t)H_\psi + tH.$$

Thus $H_\psi = H_0 \geq H_t \geq H_1 = H > 0$. Consider the family of Dirichlet problems

$$\begin{cases} Q_t[v] = \operatorname{div} \left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right) + H_t = 0 \text{ in } \Omega, & v \in C^{2,\alpha}(\bar{\Omega}) \\ v|_{\partial\Omega} = \varphi \end{cases} \quad (1)$$

where $\varphi = \psi|_L$. The set S of $t \in [0, 1]$ for which (1) has a solution is non-empty since $0 \in S$. Moreover, it is open by the implicit function theorem.

To prove that S is closed it suffices to show the existence of a uniform C^1 bound of any solution of (1). Since ψ is a supersolution for Q_t with $\varphi = \psi|_L$ and $w = 0$ is a subsolution, we have from the maximum principle and for any t that

$$0 \leq v \leq \sup \psi \leq \langle V, e \rangle$$

where $v \in C^{2,\alpha}(\bar{\Omega})$ is a solution of (1). Therefore,

$$|v|_0 \leq \langle V, e \rangle.$$

To estimate the gradient of v at the boundary it is enough to construct a local barrier from below in a neighborhood of L in $\bar{\Omega}$ since ψ is a global barrier from above with bounded gradient (see p. 333 of [7]). Then, a global barrier for the gradient follows from either Section 5 of [3] or Lemma 6 of [4].

Let η be the unit normal vector of L pointing to Ω . There is $\epsilon > 0$ such that the normal exponential map

$$E(s, y) = y + s\eta(y) \quad \text{for } s \in [0, \epsilon] \text{ and } y \in L,$$

is a diffeomorphism from $[0, \epsilon] \times L$ onto a closed neighborhood $\Lambda = E([0, \epsilon] \times L)$ of L on $\bar{\Omega}$. We take local coordinates $\{x_1, \dots, x_n\}$ on Λ where $x_1 = s = d(x)$ is the distance function to L and the remaining are local coordinates for L . We denote the corresponding coordinate vector fields by $\partial_1 = \partial_s, \partial_2, \dots, \partial_n$. Then, the metric $\sigma_{ij}dx_i dx_j$ satisfies $\sigma_{11} = 1$ and $\sigma_{1j} = 0$ if $j \geq 2$.

Choose $t \in [0, 1]$. Then $Q_t[v]$ takes the form

$$Q_t[v] = \frac{1}{A^{1/2}} \left(\sigma^{ij} - \frac{v^i v^j}{A} \right) v_{i,j} + H_t = 0$$

where we denote $A = 1 + |\nabla v|^2$.

Let $w \in C^{2,\alpha}(\Lambda)$ be given by

$$w(x) = \xi(s) + \phi(x)$$

where $\xi(s)$ is a real function to be chosen later and $\phi(x)$ is defined by extending φ to Λ as constant along the straight lines normal to L . Then, we have

$$Q_t[w] = \frac{1}{A^{1/2}} \left(\sigma^{ij} - \frac{(\xi^i + \phi^i)(\xi^j + \phi^j)}{A} \right) (\xi_{i;j} + \phi_{i;j}) + H_t$$

where

$$A = 1 + |\nabla \phi(x)|^2 + \xi_s^2(d(x)) = B + \xi_s^2.$$

From $|\nabla d| = 1$, we have $d^i d_{i;j} = 0$. Since $2\langle \nabla_{\partial d} \nabla d, \partial d \rangle = \partial d |\nabla d|^2 = 0$, we obtain

$$\Delta d|_{d=s} = \sigma^{ij} \langle \nabla_{\partial_i} \nabla d, \partial_j \rangle = -\sigma^{ij} b_{i;j}(s) = -H^s$$

where H^s denotes the mean curvature of $E(L \times \{s\})$. Then,

$$Q_t[w] \geq \frac{1}{A^{1/2}} \left(\sigma^{ij} - \frac{(\xi^i + \phi^i)(\xi^j + \phi^j)}{A} \right) \xi_{i;j} - \Lambda |\phi|_2 + H_t \quad (2)$$

where $\Lambda = 1/A^{1/2}$ is the largest eigenvalue in Ω of Q_t .

We conclude from (2) using

$$\xi^i \xi^j \xi_{i;j} = \xi_s^2 d^i d^j (\xi_{ss} d_i d_j + \xi_s d_{i;j}) = \xi_s^2 \xi_{ss},$$

$$\xi^i \phi^j \xi_{i;j} = \xi_s d^i \phi^j (\xi_{ss} d_i d_j + \xi_s d_{i;j}) = \xi_s \xi_{ss} \langle \nabla d, \nabla \phi \rangle = 0,$$

$$\phi^i \phi^j \xi_{i;j} = \phi^i \phi^j (\xi_{ss} d_i d_j + \xi_s d_{i;j}) = -\phi^i \phi^j \xi_s b_{i;j}(s)$$

and

$$\sigma^{ij} \xi_{i;j} = \Delta \xi = \xi_{ss} + \xi_s \Delta d = \xi_{ss} - \xi_s H^s$$

that

$$A^{3/2} Q_t[w] \geq B \xi_{ss} - A H^s \xi_s + \phi^i \phi^j b_{i;j}(s) \xi_s - A^{3/2} \Lambda |\phi|_2 + A^{3/2} H_t.$$

We take $\xi \in C^\infty([0, \epsilon])$ of the form

$$\xi(s) = \delta \ln(1 + \beta s)$$

for constants $\delta < 0$ and $\beta > 0$ to be determined. Thus

$$\xi_s = \frac{\delta \beta}{1 + \beta s} < 0 \quad \text{and} \quad \xi_{ss} = -\frac{1}{\delta} \xi_s^2$$

We obtain,

$$A^{3/2}Q_t[w] \geq -B\delta^{-1}\xi_s^2 - (B + \xi_s^2)H^s\xi_s + \phi^i\phi^jb_{i,j}(s)\xi_s + (B + \xi_s^2)^{3/2}H_t - (B + \xi_s^2)|\phi|_2.$$

Since $B = 1 + |\nabla\phi(x)|^2 \geq 1$ and $\xi_s < 0$, we have

$$A^{3/2}Q_t[w] \geq (H^s + H_t)(-\xi_s)^3 - (B\delta^{-1} + |\phi|_2)\xi_s^2 + C\xi_s + D$$

where the functions C and D depend only on the metric of Λ and on the function φ and its derivatives. Since $H^0 = H_L$ and $H_t \geq H \geq -H_L$, it follows that

$$H^0 + H_t = H_L + H_t \geq H_L + H \geq 0.$$

Therefore, at points of L we have that

$$\lim_{s \rightarrow 0} A^{3/2}Q_t(w) \geq -(B\delta^{-1} + |\phi|_2)(\delta\beta)^2 + C\delta\beta + D.$$

We choose δ such that $B/\delta + |\phi|_2 < 0$. Then, there is β independent of t and large enough such that $A^{3/2}Q_t[w] \geq 1$ in the neighborhood $\Lambda' = E(L \times [0, \epsilon_1])$ of L on $\bar{\Omega}$ for some $\epsilon_1 \in (0, \epsilon]$.

To assure that w is a local barrier from below for Q_t in Λ' we have to guarantee that

$$w|_{\partial\Lambda'} \leq v|_{\partial\Lambda'}. \quad (3)$$

At L we have that $w = \varphi$ and (3) is trivially satisfied. At the other component of the boundary of Λ' condition (3) is verified if

$$w(x, \epsilon_1) = \delta \ln(1 + \epsilon_1\beta) + \varphi(x) \leq \delta \ln(1 + \epsilon_1\beta) + \langle V, e \rangle \leq 0,$$

and this condition is clearly satisfied by choosing β large enough.

We have proved that w is a barrier from below of Q_t in Λ' by appropriate choices of δ and β independent of t . It follows that the C^1 -norm of w in Ω can be estimate by a bound which does not depend on t . Since $w \leq v \leq \psi$ in Λ and

$$v|_L = \varphi = w|_L = \psi|_L$$

by the gradient maximum principle (cf. Theorem 15.1 in [7]), we have

$$\max_{\Omega} |\nabla v| = \max_L |\nabla v| \leq \max\{\max_L |\nabla w|, \max_{\Omega} |\nabla \psi|\} \leq M$$

where $M > 0$ does not depend on t . Finally, we take $C = \max\{\langle V, e \rangle, M\}$. This proves the existence of a priori C^1 estimates for the solutions of (1) which implies that S is closed and concludes the proof of the theorem in the case of finite V .

We now use the previous case to prove the theorem for $V = \infty$, that is, to obtain Serrin's result. Given $n \in \mathbb{N}$ and setting

$$H_n = \frac{n}{n+1}H,$$

there is V_n high enough such that the cone

$$J_n = \{tV_n + (1-t)p : p \in \Gamma \text{ and } t \in [0, 1]\},$$

is an H_n -cone. Moreover, since $H_L \geq H \geq H_n$, we may use the previous case to assert the existence of a solution $u_n \in C^{2,\alpha}(\bar{\Omega})$ of $Q_{H_n} = 0$ in Ω such that $u_n|_{\partial\Omega} = \varphi$. We have the well-known height estimates

$$|u_n|_0 \leq \frac{2}{H_n} \leq \frac{4}{H}.$$

Moreover, a similar local barrier used above to estimate the gradient of u_n from below can be used to estimate the gradient of u_n from above and, as before, this provides uniform (that is, no depending on n) C^1 estimates of the sequence u_n . From linear elliptic PDE theory this guarantees C^2 compactness of $\{u_n\}$. Therefore, a subsequence of u_n converges to a solution $u \in C^2$ of $Q_H = 0$ such that $u|_{\partial\Omega} = \varphi$. Finally, PDE regularity implies that $u \in C^{2,\alpha}(\bar{\Omega})$. ■

Proof of Theorem 2. Due to Serrin's result for the C^0 case (Theorem 16.11 in [7]) it suffices to consider the case of V finite. Without loss of generality, we assume that the domain enclosed by γ contains the origin of \mathbb{R}^n . Clearly, the domain D_{γ_k} enclosed by $\gamma_k = (1 + 1/k)\gamma$ contains $\bar{\Omega}$ for any k . Set $V_k = (1 + 1/k)V$. Then, the cone

$$K_k = K_{V_k}(\gamma_k) = (1 + 1/k)K_V(\gamma)$$

is an H_k -cone for $H_k = (1 + 1/k)H$. Since $H_k > H$, we thus have

$$H_L \geq -H > -H_k. \tag{4}$$

Let $\psi_k \in C^0(\bar{\Omega})$ be the function whose graph is

$$S_k := \bar{\Omega} \times [0, +\infty) \cap K_k$$

and $\varphi_k \in C^{2,\alpha}(L)$ the function which graph is ∂S_k . We apply Perron's method on $\bar{\Omega}$ to the constant mean curvature hypersurface elliptic PDE

$$Q_{H_k}[v] = \operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) + H_k = 0 \tag{5}$$

in Ω with boundary condition $\psi_k|_L = \varphi_k$.

It is clear that ψ_k is a supersolution for (5). On the other hand, we have from (4) that the mean curvature of the cylinder $L \times \mathbb{R}$ is strictly smaller than $-H_k$. Then, there is $z_0 \gg 0$ such that the cone

$$J_k = \{t(V - z_0 e) + (1 - t)p : p \in \partial S_k \text{ and } t \in [0, 1]\}$$

has mean curvature smaller than $-H_k$. If J_k is the graph of the function $\chi_k \in C^0(\bar{\Omega})$ then χ_k is a subsolution for Q_{H_k} in $\bar{\Omega}$ such that $\chi_k|_L = \varphi_k$. It follows from Perron's method that

$$v_k(x) = \sup \{ \sigma(x) : \sigma \in C^0(\bar{\Omega}) \text{ is a subsolution for } Q_{H_k} \text{ in } \Omega, \sigma|_L = \varphi_k \}, \quad x \in \bar{\Omega}$$

is in $C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$ and is a solution of $Q_{H_k} = 0$ in Ω such that $v_k|_L = \varphi_k$.

We have that $v_k \in C^\infty(\Omega)$ by interior regularity. Moreover, since $\chi_k \leq u \leq \psi_k$ for all k and $\varphi_k = \chi_k|_{\partial\Omega} = \psi_k|_{\partial\Omega}$ converges to φ as $k \rightarrow \infty$, it follows that u extends continuously to $\bar{\Omega}$ and $u|_{\partial\Omega} = \varphi$. ■

Remark 3. It is clear that we may replace in Theorems 1 and 2 the H -cone by any supersolution of Q_H . Observe that H -cone like supersolutions can be constructed as follows: Consider a star-shaped compact embedded hypersurface $\gamma \subset \mathbb{R}^n$ with respect to a point $O \in D$ and take a point $V \in \mathbb{R}_+^n \cup \{\infty\}$ in the half straight line r orthogonal to \mathbb{R}^n with origin O . Then, in each hyperplane R containing r choose a smooth simple curve α_R starting in V (asymptotic to r if $V = \infty$) finishing in γ and having a single valued projection on \mathbb{R}^n . Moreover, the α_R 's are chosen to depend smoothly on R so that their union as R varies constitute a graph G in \mathbb{R}^{n+1} which is smooth in $G \setminus \{V\}$ (in G if $V = \infty$). Given $H \geq 0$, it is clear that we may construct such a G to be an H -supersolution for Q_H on a domain containing O .

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